

Finite temperature spin dynamics in a perturbed quantum critical Ising chain with an E_8 symmetry

Jianda Wu,¹ Márton Kormos,^{1,2} and Qimiao Si¹

¹*Department of Physics & Astronomy, Rice University, Houston, Texas 77005, USA*

²*MTA-BME “Momentum” Statistical Field Theory Research Group, 1111 Budapest, Budafoki út 8, Hungary*

Abstract

A spectrum exhibiting E_8 symmetry is expected to arise when a small longitudinal field is introduced in the transverse-field Ising chain at its quantum critical point. Evidence for this spectrum has recently come from neutron scattering measurements in cobalt niobate, a quasi-one-dimensional Ising ferromagnet. Unlike its zero-temperature counterpart, the finite-temperature dynamics of the model has not yet been determined. We study the dynamical spin structure factor of the model at low frequencies and nonzero temperatures, using the form factor method. Its frequency dependence is singular, but differs from the diffusion form. The temperature dependence of the nuclear magnetic resonance (NMR) relaxation rate has an activated form, whose prefactor we also determine. We propose NMR experiments as a means to further test the applicability of the E_8 description for CoNb_2O_6 .

PACS numbers: 71.10.Hf, 73.43.Nq, 71.10.-w.

Introduction.— Quantum criticality is a subject of extensive interest in various contexts [1, 2]. These range from correlated-electron bulk materials, which can be tuned to the border of magnetism, to systems in low dimensions, where quantum fluctuations are enhanced. The collective fluctuations of a quantum critical point (QCP) often lead to unusual properties. Even in equilibrium, the statics and dynamics are mixed at a QCP. This gives rise to dynamical scaling, while also making it difficult to calculate the fluctuation spectrum. The latter is especially so for the dynamics at nonzero temperatures ($T > 0$) in the “quantum relaxational” regime, which corresponds to small frequencies ($\omega \ll k_B T/\hbar$) or long times. Indeed, even for the canonical QCP of a transverse-field Ising model in one dimension, it has been challenging to calculate such real-frequency dynamics [3, 4].

We are interested here in the one dimensional transverse field Ising model in the presence of a small longitudinal field. The transverse-field-induced QCP in the absence of a longitudinal field [5] has an emergent conformal invariance in the scaling limit [6]. When a small longitudinal field is turned on at the QCP, the excitation spectrum becomes discrete at low energies. The perturbed conformal field theory [7] provided evidence that the discrete spectrum corresponds to eight particles whose masses (of relativistic dispersions) have definite ratios, and the corresponding integrable quantum field theory has a celebrated E_8 symmetry. The first two particles correspond to bound states that are well below the continuum part of the spectrum. Recently neutron scattering measurements have been carried out in a ferromagnetic cobalt niobate CoNb_2O_6 , whose Co^{2+} are coupled in a quasi-1D way; the experiment identified two excitations whose energy ratios are close to the predicted value, the golden ratio [8].

In this letter, we study the low-frequency dynamical spin structure factor at finite temperatures using the

form factor method [9]. From theoretical perspective, our calculation provides an illustrative setting to determine the dynamics in the quantum-relaxational regime. For the E_8 model, the dynamics at finite temperatures have not been systematically studied. From the perspective of the material CoNb_2O_6 , our study determines the temperature dependence of the NMR relaxation rate. Our results suggest that NMR experiments at low temperatures provide a concrete and independent test on the the E_8 description for CoNb_2O_6 . We note that a numerical analysis of a generalized transverse-field Ising chain suggests that the E_8 description survives suitable generalizations of the interactions beyond the nearest-neighbor ferromagnetic coupling [10].

The Model.— Consider the Hamiltonian

$$H_Z = -J \left(\sum_i \sigma_i^z \sigma_{i+1}^z + g \sum_i \sigma_i^x + h_z \sum_i \sigma_i^z \right) \quad (1)$$

where σ_i^x and σ_i^z are the Pauli matrices associated with the spin components $S^\mu = \sigma^\mu/2$, ($\mu = x, y, z$), i marks a site position, and g and h_z are respectively the transverse and longitudinal fields in unit of the nearest-neighbor ferromagnetic exchange coupling J among the longitudinal (z) components of the spins. In the absence of the longitudinal field ($h_z = 0$) the system undergoes a quantum phase transition when the transverse field is tuned across its critical value $g = g_c = 1$ [5]. As is well known, the QCP is described by a 1 + 1-dimensional conformal field theory (CFT) with a central charge 1/2 [6]. More surprising is what happens when a small longitudinal field h is introduced at the QCP $g = g_c$. Here, the model in the scaling limit can be described by an integrable quantum field theory. This E_8 model [7, 9] corresponds to the action

$$\mathcal{A}_{E8} = \mathcal{A}_{c=1/2} + h \int dx d\tau \sigma(\tau, x). \quad (2)$$

where τ is the imaginary time. Here, $\mathcal{A}_{c=1/2}$ stands for the action of the two dimensional CFT with central charge $1/2$, h has scaling dimension $15/8$, and $\sigma(x)$ is a primary field with scaling dimension $1/8$. This describes a scattering theory of eight massive particles, which we will denote by a, b, c, d, e, f, g, h from the lightest to the heaviest. The mass of the lightest particle, Δ_a , scales with the longitudinal field as $\Delta_a \approx 4.405 |h|^{8/15}$ [11]. The mass of the second lightest particle Δ_b is Δ_a multiplied by the golden ratio $(\sqrt{5} + 1)/2$. These two particles are clearly separated from the two-particle continuum, which appears at energies above $2\Delta_a$.

Local dynamics and NMR relaxation rate.— We focus on the local dynamical structure factor (DSF) of the E_8 model in the low frequency and low temperature limit: $\omega \ll \Delta_a$ and $T \ll \Delta_a$ (hereafter we set $\hbar = 1$ and $k_B = 1$).

A useful means to probe the local DSF is via NMR. The NMR relaxation rate is given by [12]

$$\frac{1}{T_1^\alpha} = \frac{1}{2N} A^2 \sum_{\beta}' S_{\beta\beta}(\omega_0). \quad (3)$$

Here, α and β label the principal axes, and the primed summation is over the principal axes perpendicular to the field orientation α ; T_1^α is the spin-lattice relaxation time, N is the number of ions per unit cell, and ω_0 is the nuclear resonance frequency. In addition, A describes the hyperfine coupling between the spins of the a nucleus and the electrons; while this coupling depends on the wavevector \mathbf{q} , the dependence is generically smooth and we will take it as a constant. We will consider the static field of the NMR setup to be the transverse field, $\alpha = x$. Correspondingly, the local DSF of interest to NMR is given by $S_{zz}(\omega_0) + S_{yy}(\omega_0)$. As shown in the supplementary materials [25], for the model we consider,

$$S_{yy}(\omega) = \omega^2 S_{zz}(\omega)/4 \quad (4)$$

Thus, in the low-frequency regime of interest here, $S_{yy}(\omega)$ is negligible compared with $S_{zz}(\omega)$. In the following, we will therefore only consider S_{zz} .

We now turn to the calculation of $S_{zz}(\omega)$ through a systematic form factor expansion. Because the excitation spectrum has a gap, we expect that the leading contributions in the low temperature and low frequency limit come from those associated with the few particle states of the light particles. Indeed, we show below that the dominant contribution comes from the two 1-particle states of the lightest particle, which we calculate analytically. The conclusion is confirmed by a numerical calculation for contributions that extend to higher orders.

The form factor series.— Integrable field theory techniques made possible the analytic calculation of matrix elements of local observables in the asymptotic scattering state basis, called *form factors*. The asymptotic

states are eigenstates of the energy and momentum operators. In terms of the rapidities $\{\theta_i\}$ of the particles, the energy and momentum eigenvalues of the eigenstate $|\theta_1^{\alpha_1}, \dots, \theta_n^{\alpha_n}\rangle$ (with $\{\alpha_1, \dots, \alpha_n\}$ marking different types of particles) are

$$E = \sum_{i=1}^n \Delta_{\alpha_i} \cosh(\theta_i), \quad (5)$$

$$P = \sum_{i=1}^n \Delta_{\alpha_i} \sinh(\theta_i). \quad (6)$$

We denote by $F_n^\sigma(\theta_1^{\alpha_1}, \dots, \theta_n^{\alpha_n})$ the form factors of the primary field $\sigma(x, t)$ in the E_8 model (c.f. Eq. (2)) between the vacuum and an n -particle asymptotic state,

$$F_n^\sigma(\theta_1^{\alpha_1}, \dots, \theta_n^{\alpha_n}) = \langle 0 | \sigma(0, 0) | \theta_1^{\alpha_1}, \dots, \theta_n^{\alpha_n} \rangle. \quad (7)$$

The few-particle form factors are explicitly known [14–16] and have been used to calculate the static spin-spin correlations of the E_8 model in the ground state [14, 15]. Here we study the finite-temperature dynamics by a low-temperature expansion series for integrable field theory [17, 18], using a finite-volume regularization [18].

The finite temperature two-point correlation function is given by

$$C(t, x) = \text{Tr} \left[\frac{e^{-H/T}}{\mathcal{Z}} \mathcal{O}(t, x) \mathcal{O}^\dagger(0, 0) \right], \quad (8)$$

where $\mathcal{Z} = \text{Tr} e^{-H/T}$ is the partition function, and we are interested in the local observable operator $\mathcal{O}(t, x) = \sigma(t, x)$. The corresponding DSF is

$$S(\omega, q) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dt C(t, x) e^{i\omega t - iqx}, \quad (9)$$

We insert the complete set of asymptotic states between the operators, yielding a double sum, $C(t, x) = \mathcal{Z}^{-1} \sum_{r,s} C_{r,s}(t, x)$, where

$$C_{r,s}(t, x) = \sum_{\{\alpha_j\}, \{\alpha'_k\}} \int \frac{d\theta_1 \dots d\theta_r}{(2\pi)^r r!} \int \frac{d\theta'_1 \dots d\theta'_s}{(2\pi)^s s!} e^{-\beta E_r} e^{-it(E_s - E_r)} e^{-i(P_r - P_s)x} \left| \langle \theta_1^{\alpha_1} \dots \theta_r^{\alpha_r} | \mathcal{O} | \theta_1^{\alpha'_1} \dots \theta_s^{\alpha'_s} \rangle \right|^2, \quad (10)$$

with $E_n = \sum_{i=1}^n \Delta_{\alpha_i} \cosh \theta_i$, $P_n = \sum_{i=1}^n \Delta_{\alpha_i} \sinh \theta_i$.

We use the same set of states to write the partition function as $\mathcal{Z} = \sum_{n=0}^{\infty} \mathcal{Z}_n$ where

$$\mathcal{Z}_n = \sum_{\{\alpha_j\}} \int \frac{d\theta_1 \dots d\theta_n}{(2\pi)^n n!} e^{-\beta E_n} \langle \theta_1^{\alpha_1} \dots \theta_n^{\alpha_n} | \theta_1^{\alpha_1} \dots \theta_n^{\alpha_n} \rangle. \quad (11)$$

In infinite volume all the \mathcal{Z}_n 's contain singularities associated with the scalar product of two momentum eigenstates with identical rapidities. Similarly, for the observables we are calculating, $C_{r,s}$ also diverge due to the kinematical poles of the form factors whenever two rapidities

in the two sets coincide, $\theta_i = \theta'_j$ [17]. However, the double sums can be re-organized such that the aforementioned singularities cancel each other [18],

$$C(t, x) = \sum_{r,s=0}^{\infty} D_{r,s}(t, x), \quad (12)$$

where

$$D_{0,s} = C_{0,s}, \quad (13)$$

$$D_{1,s} = C_{1,s} - \mathcal{Z}_1 C_{0,s-1}, \quad (14)$$

$$D_{2,s} = C_{2,s} - \mathcal{Z}_1 C_{1,s-1} + (\mathcal{Z}_1^2 - \mathcal{Z}_2) C_{0,s-2}, \quad (15)$$

$$\dots \text{etc.} \quad (16)$$

The natural small parameter in the series (12) is $e^{-\Delta_a/T}$. At low frequencies, the energy conserving Dirac-deltas in the Fourier transform Eq. (9) force the two states appearing in the form factors to have nearly equal energy, $E_r = \omega + E_s$. The magnitude of the Boltzmann factor is then set by the sum of the masses in the “heavier” state, i.e.,

$$D_{r,s} \sim \exp \left\{ -\frac{1}{T} \max \left[\sum_{i=1}^r \Delta_i, \sum_{i=1}^s \Delta_i \right] \right\}. \quad (17)$$

Thus, in the regime of interest ($T/\Delta_a \ll 1$ and $\omega/\Delta_a \ll 1$), the expansion series in Eq. (12) is a good perturbation series. In this regime, we can safely truncate the series beyond the terms up to the order of $e^{-2\Delta_a/T}$. Simple counting implies that we only need $D_{0,1}, D_{1,0}, D_{0,2}, D_{2,0}, D_{1,1}, D_{1,2} + D_{2,1}, D_{2,2}$ with lightest particles, which we now determine. We also note that the series for the two-point correlator *per se* contain a $\delta(\omega)$ piece, which are however absent in the connected correlation function of interest here [25].

Leading contributions.— $D_{0,1}$ is the channel between vacuum and one-particle asymptotic in state, and is equal to $C_{0,1}$ from Eq. (13). The corresponding contribution

to DSF is

$$S_{0,1}(\omega, q) = 2\pi |F_1^\sigma|^2 \int d\theta \delta(q - \Delta_1 \sinh \theta) \delta(\omega - \Delta_1 \cosh \theta), \quad (18)$$

where Δ_1 is the mass of a single particle state, and the one particle form factor $F_1^\sigma(\theta)$ is rapidity independent[14]. Since $\cosh \theta \geq 1$ always holds, for the parameter regime $\omega < \Delta_a$ the terms $S_{0,1}$ and $S_{1,0}$ do not contribute. Similarly, the $D_{0,s}$ and $D_{r,0}$ terms for general r and s also vanish.

The first non-trivial contribution is given by connected parts in $D_{1,1}$, i.e. the term coming from the 1-particle – 1-particle form factors, for which we obtain [25]

$$S_{1,1}(\omega, q) = \frac{|F_2^\sigma(\alpha + i\pi, 0)|^2 (e^{-\beta\Delta_1 \cosh \theta_+} + e^{-\beta\Delta_1 \cosh \theta_-})}{\Delta_1 \Delta_2 |\sinh \alpha|}, \quad (19)$$

where Δ_1 and Δ_2 are the masses of the 1-particle states, $\alpha = \text{arccosh}[(\Delta_1^2 + \Delta_2^2 - (\omega^2 - q^2))/(2\Delta_1 \Delta_2)]$ and $\cosh \theta_{\pm} = [\omega(\Delta_1^2 - \Delta_2^2 + \omega^2 - q^2) \pm 2q\Delta_1 \Delta_2 \sinh \alpha]/[2\Delta_1(q^2 - \omega^2)]$; hereafter the symbols that denote the types of particles in the form factor are dropped for notational convenience [Eq. (7)].

The corresponding local DSF is $S_{1,1}(\omega) = \int_{-\infty}^{\infty} S_{1,1}(q, \omega) dq$. Eq. (17) implies that, up to $e^{-2\Delta_a/T}$, we need only to consider the channels $a - a$, $b - b$ and $c - c$, as well as $a - b$, $a - c$, $b - c$. When $\Delta_1 = \Delta_2 = \Delta_i$ ($i = a, \dots, h$),

$$S_{1,1}(\omega)|_{\Delta_1=\Delta_2=\Delta_i} = \int_{\omega}^{\infty} f(q, \omega) e^{-\frac{\Delta_i}{T} g(q, \omega)} dq \quad (20)$$

with $f(q, \omega) = \frac{2}{\Delta_i^2} |F_2^\sigma(\alpha + i\pi, 0)|^2 / |\sinh \alpha|$ and $g(q, \omega) = -\frac{\omega}{2\Delta_i} + \frac{q}{2\Delta_i} \sqrt{1 + \frac{4\Delta_i^2}{q^2 - \omega^2}}$. We can expand the result for small ω . With the details given in the supplementary material [25], we find the result to leading order:

$$S_{1,1}(\omega)|_{\Delta_1=\Delta_2=\Delta_a} \approx \begin{cases} \frac{2|F_2^\sigma(i\pi, 0)|^2}{\Delta_a} e^{-\Delta_a/T} \left\{ \ln \frac{4T}{\omega} - \gamma_E + \dots \right\} & (\omega \ll T \ll \Delta_a) \\ \frac{2|F_2^\sigma(i\pi, 0)|^2}{\Delta_a} e^{-\Delta_a/T} \left\{ \sqrt{\frac{\pi T}{\omega}} - \frac{\sqrt{\pi}}{4} \left(\frac{T}{\omega}\right)^{3/2} + \dots \right\} & (T \ll \omega \ll \Delta_a) \end{cases}, \quad (21)$$

where γ_E is the Euler constant. (The same form applies to the contributions by the other particles b, \dots, h , which are suppressed by their thermal factors.) In deriving this expression, we have replaced $\alpha(\omega, q)$ by $\alpha(\omega = 0, q = 0)$. This is because the dominant contribution comes from the minimum of the energy dispersion at small momentum; it is well supported by the numerical calculation

carried out without this replacement (see below).

We observe that the finite-T local DSF diverges logarithmically as $\omega \rightarrow 0$. This divergence differs from the diffusion form [13] of inverse square root; this is reasonable given that the total S_z is not conserved here. When $\Delta_1 \neq \Delta_2$, the denominator in the integrand of Eq. (19) does not have any singularity so there will be no diver-

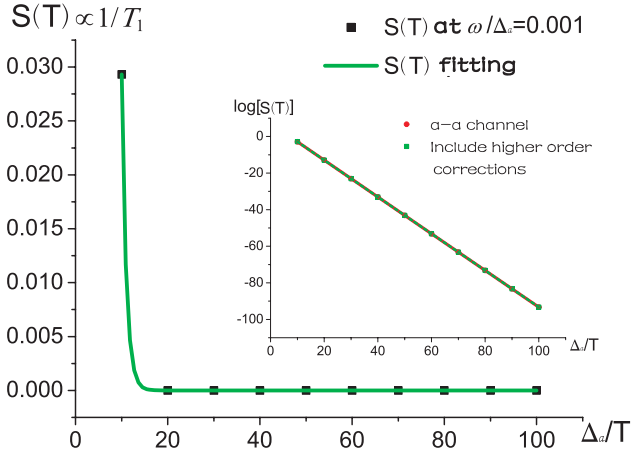


FIG. 1: The NMR relaxation rate as a function of temperature. The frequency is chosen to be $\omega/\Delta_a = 0.001$. The temperature dependence is well described by $S(T) = 631e^{-\Delta_a/T}$. The inset picture shows that channels other than $a-a$ give negligible contributions.

gence.

Next, we consider $D_{1,2} + D_{2,1}$, the terms with a one-particle and a two-particle state. Up to the order $\mathcal{O}(e^{-2\Delta_a/T})$, we focus on the case when all three particles are the lightest a particle (the other channels $aa-b$ and $aa-c$ are expected to behave similarly), which we find to be [25],

$$S_{(1,2)+(2,1)}(\omega, q) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\theta e^{-\beta\Delta_a \cosh \theta}}{\sqrt{(f(\tilde{\omega}, \tilde{q}, \theta) - 1)^2 - 1}} \cdot F_3^\sigma(\theta + i\pi, \ln x_+, \ln x_-) F_3^\sigma(\theta + i\pi, \ln x_-, \ln x_+), \quad (22)$$

where

$$x_{\pm} = \frac{1}{2}(\tilde{\omega} + \cosh \theta + \tilde{q} + \sinh \theta) \left(1 \pm 2\sqrt{1 - 2/f(\tilde{\omega}, \tilde{q}, \theta)}\right), \quad (23)$$

and $f(\tilde{\omega}, \tilde{q}, \theta) = [(\tilde{\omega} + \cosh \theta)^2 - (\tilde{q} + \sinh \theta)^2]/2$ with $\tilde{\omega} = \omega/\Delta_a$ and $\tilde{q} = q/\Delta_a$. Our analysis [25] shows no contributions from the range $\tilde{\omega} > \tilde{q} \geq 0$, where $\cosh \theta \sim 1/\tilde{\omega} \gg 1$. In the range $\tilde{\omega} \leq \tilde{q}$, we have $\cosh \theta \gtrsim 2 - \tilde{\omega}$, indicating there exists a small region of \tilde{q} where $\cosh \theta$ is slightly smaller than 2. This contribution is expected to be small, and we confirm this by including the channels $D_{1,2} + D_{2,1}$ in our numerical calculation shown below.

For connected parts in $D_{2,2}$, a similar Jacobian will appear as in the calculation of the equal mass case of Eq. (S28), and we will encounter the same logarithmic divergence in the frequency dependence. We find no singular terms beyond the logarithmic divergence [25]. This contribution is therefore suppressed by the thermal weight $e^{-2\Delta_a/T}$. Low-frequency divergences are also expected to come from the D_{nn} terms (at $n > 2$) with

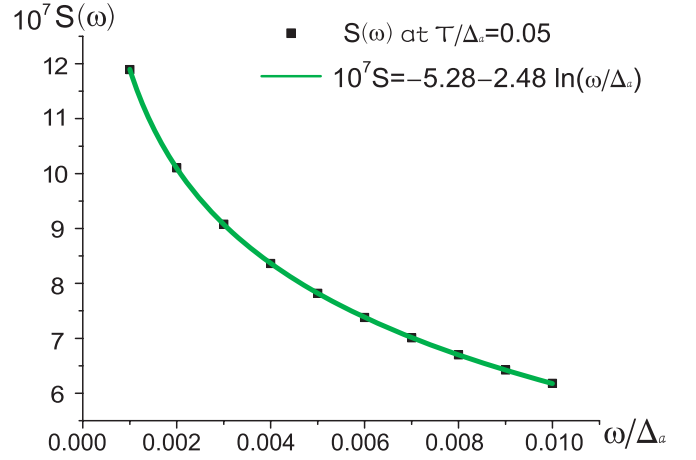


FIG. 2: The local dynamical structure factor as a function of frequency at a fixed temperature $T/\Delta_a = 0.05$. The ω -dependence is well described by $10^7 S(\omega) = -5.28 + 2.48 \ln(\Delta_a/\omega)$.

particles of the same mass in the two asymptotic states of the form factors. The fact that D_{22} with the same particle does not contain singularities stronger than $\ln \omega$ is a strong indication that none of the higher terms in the series will give a stronger (e.g. power-law) singularity. We conjecture that the D_{nn} terms at $n > 2$ have a similar logarithmic singularity in the frequency dependence, and they are then also negligible compared to D_{11} due to the stronger thermal suppression factor.

Numerical analysis.— Fig. 1 shows the results and fit for the NMR relaxation rate as a function of temperature in the range $\Delta_a/T \in [10, 100]$ at a fixed low frequency $\omega/\Delta_a = 0.001$ appropriate for the NMR experiments (satisfying $\omega \ll T$). The fitting function $S(T) = 631e^{-\Delta_a/T}$ indicates that the behavior of relaxation rate at low frequency and low temperature region is dominated by the contribution from the $a-a$ channel, as clearly shown in the inset to Fig. 1. The prefactor 631 compares well with the analytical expression associated with $S_{1,1}$ of the lightest a -particle: since $2|F_2^\sigma(i\pi, 0)|_{\Delta_1=\Delta_2=\Delta_a}^2 \approx 130$.

We also study the frequency dependence of the local DSF at fixed temperatures for $T, \omega \ll \Delta_a$. Fig. 2 shows the result at a fixed $T/\Delta_a = 0.05$ with ω/Δ_a ranging from 0.001 to 0.01 (satisfying $\omega \ll T$). It is well fitted as $10^7 S(\omega) = -5.28 - 2.48 \ln(\omega/\Delta_a)$, which is in accordance with the asymptotic form Eq. (21).

Discussion.— We conclude that the temperature dependence of the NMR relaxation rate is given by

$$\frac{1}{T_1} \approx \frac{c^2 b}{2N} A^2 e^{-\Delta_a/T}; \quad \Delta_a \approx 4.405 |h|^{8/15}. \quad (24)$$

In the prefactor, $c \approx 0.783$ converts the σ field of the continuum theory to the lattice spin [14], and $b \approx \ln(4T/\omega_0) - \gamma_E$.

We next consider the implications of our results for CoNb_2O_6 . The neutron scattering experiments provided evidence for the two lightest particles of the E_8 spectrum [8]. This has been understood by considering the effect of the inter-chain coupling in the three-dimensionally ordered state as inducing a longitudinal field [8, 21]. In order to further test the E_8 description, measuring the temperature dependence of the spin dynamics would be invaluable. Our study here provides a concrete prediction of the temperature-dependence of the NMR relaxation rate in the E_8 model, which can be used for the desired further test. During the final stage of writing the present manuscript, an NMR experiment in CoNb_2O_6 has been reported in the higher-temperature quantum critical regime [22]; measurements of the NMR relaxation rate at the lower-temperature E_8 regime should therefore be feasible.

To summarize, we have determined the local dynamical spin structure factor of the perturbed quantum-critical Ising chain at temperatures and frequencies that are small compared to the mass of the lightest E_8 particle. The frequency dependence shows a logarithmic singularity. Our calculation yields a concrete prediction for the temperature dependence of the NMR relaxation rate, which we have suggested as a means to further test the E_8 description of the spin dynamics in CoNb_2O_6 .

ACKNOWLEDGEMENT

We thank R. Coldea, F.H.L.Essler and J. H.H. Perk for useful discussions. J.W. and Q.S. acknowledge the support provided in part by the NSF Grant No. DMR-130953 and the Robert A. Welch Foundation Grant No. C-1411, and M.K. acknowledges that by the Marie Curie IIF Grant PIIF-GA-2012- 330076. Q.S. also acknowledges the support of the Alexander von Humboldt Foundation, and the hospitality of the the Karlsruhe Institute of Technology and the Institute of Physics of Chinese Academy of Sciences.

- [8] R. Coldea, D. A. Tennant, E. M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smeibidl, K. Kiefer, Science 327, 177 (2010).
- [9] G. Mussardo, *Statistical Field Theory* (Oxford University Press, 2010).
- [10] J. A. Kjäll, F. Pollmann, and J. E. Moore, Phys. Rev. B 83, 020407(R) (2011).
- [11] A.I. B. Zamolodchikov, Int. J. Mod. Phys. A10, 1125 (1995).
- [12] T. Moriya, Prog. Theor. Phys. 2, 371 (1962).
- [13] G. F. Giuliani and G. Vignale, *Quantum Theory of the Electron Liquid*, 2005, Cambridge University Press.
- [14] G. Delfino and G. Mussardo, Nucl. Phys. B 455, 724 (1995).
- [15] G. Delfino and P. Simonetti, Phys. Lett. B 383, 450 (1996).
- [16] G. Delfino, P. Grinza and G. Mussardo, Nucl. Phys. B 737, 291 (2006).
- [17] F. H. L. Essler and R. M. Konik, J. Stat. Mech. 9, P09018 (2009).
- [18] B. Pozsgay and G. Takacs, J. Stat. Mech. 11, P11012 (2010).
- [19] L.M.Szécsényi and G. Takács, J. Stat. Mech 12, P12002 (2012).
- [20] B. Pozsgay and G. Takács, Nucl. Phys. B 788, 209 (2008).
- [21] S. B. Lee, R. K. Kaul, and L. Balents, Nat. Phys. 6, 702 (2010).
- [22] A. W. Kinross et al., arXiv: 1401.6917v1 (2014).
- [23] Zhuxi Wang and Dunren Guo, *Introduction to Special Functions*, 2000, Peking University Press.
- [24] <http://www.sissa.it/~delfino/isingff.html>.
- [25] J. Wu, M. Kormos, and Q. Si, Supplemental Material.

-
- [1] Special issue: Quantum Phase Transitions. J. Low Temp. Phys. **161**, 1-323 (2010).
 - [2] S. Sachdev, *Quantum Phase Transitions*, Cambridge University Press (2011).
 - [3] P. Deift and X. Zhou, Singular Limits of Dispersive Waves 320, 183 (1994).
 - [4] S. Sachdev and A. P. Young, Phys. Rev. Lett 78, 2220 (1997).
 - [5] P. Pfeuty, Ann. of Phys. 57, 79(1970).
 - [6] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B 241, 333(1984).
 - [7] A. B. Zamolodchikov, Int. J. Mod. Phys. A4, 4235(1989).

Supplementary Material — Finite temperature spin dynamics in a perturbed quantum critical Ising chain with an E_8 symmetry

Jinda Wu, Márton Kormos and Qimiao Si

Derivation of $\chi_{yy}(x, t)$

The Hamiltonian of one dimensional transverse field Ising model with a longitudinal field can be expressed as

$$H = -J \left(\sum_i \sigma_i^z \sigma_{i+1}^z + g \sum_i \sigma_i^x + h_z \sum_i \sigma_i^z \right) \quad (S1)$$

where J , g and h_z have the same meaning as in the main text. Consider $C(i, j, t, T) = \langle \sigma_i^z(t) \sigma_j^z(0) \rangle_T$, where $\langle \cdots \rangle_T$ denotes thermal averaging. We have

$$\frac{\partial C(i, j, t, T)}{\partial t} = -Jgi \langle e^{iHt} [\sigma_i^x(0), \sigma_i^z(0)] e^{-iHt} \sigma_i^z(0) \rangle = -2Jg \langle \sigma_i^y(t) \sigma_j^z(0) \rangle_T = -2Jg \langle \sigma_i^y(0) \sigma_j^z(-t) \rangle_T \quad (S2)$$

and

$$\frac{\partial^2 C(i, j, t, T)}{\partial t^2} = (-Jg) 2 \frac{\partial \langle \sigma_i^y(0) \sigma_j^z(-t) \rangle_T}{\partial t} = (-Jg)^2 (-4) \langle \sigma_i^y(t) \sigma_j^y(0) \rangle_T = -4(Jg)^2 \langle \sigma_i^y(t) \sigma_j^y(0) \rangle_T \quad (S3)$$

Recall the definition of linear response $\chi_{ij,T}^{\alpha\alpha}$, $\alpha = x, y, z$,

$$\chi_{ij,T}^{\alpha\alpha} = -i\theta(t) \langle [\sigma_i^\alpha(t), \sigma_j^\alpha(0)] \rangle_T \quad (S4)$$

We have

$$\chi_T^{yy}(x, t) = -\frac{1}{4(gJ)^2} \frac{\partial^2 \chi_T^{zz}(x, t)}{\partial t^2} \quad (S5)$$

Then

$$\chi_T^{yy}(\omega) = \frac{\omega^2}{4(gJ)^2} \chi_T^{zz}(\omega) \quad (S6)$$

Relevant Form Factors used in the Main Text

The main text considered two- and three-particle form factors of the E_8 model. The relevant two-particle form factors are known in the literature[14, 15]. Here, for completeness, we present their detailed expressions, where “ n ” in F_n^σ is explicitly written as types of particles it contains.

$$F_{aa}^\sigma(\theta_1, \theta_2) = \{c_{11}^0 + c_{11}^1 \cosh(\theta_1 - \theta_2)\} \left\{ -i \sinh\left(\frac{\theta_1 - \theta_2}{2}\right) \right\} \frac{T_{2/3}(\theta_1 - \theta_2) T_{2/5}(\theta_1 - \theta_2) T_{1/15}(\theta_1 - \theta_2)}{P_{2/3}(\theta_1 - \theta_2) P_{2/5}(\theta_1 - \theta_2) P_{1/15}(\theta_1 - \theta_2)} \quad (S7)$$

$$F_{bb}^\sigma(\theta_1, \theta_2) = \{c_{22}^0 + c_{22}^1 \cosh(\theta_1 - \theta_2) + c_{22}^2 \cosh^2(\theta_1 - \theta_2) + c_{22}^3 \cosh^3(\theta_1 - \theta_2)\} \left\{ -i \sinh\left(\frac{\theta_1 - \theta_2}{2}\right) \right\} \cdot \frac{T_{4/5}(\theta_1 - \theta_2) T_{2/3}(\theta_1 - \theta_2) T_{7/15}(\theta_1 - \theta_2) T_{4/15}(\theta_1 - \theta_2) T_{1/15}(\theta_1 - \theta_2) (T_{2/5}(\theta_1 - \theta_2))^2}{P_{4/5}(\theta_1 - \theta_2) P_{2/3}(\theta_1 - \theta_2) P_{7/15}(\theta_1 - \theta_2) P_{4/15}(\theta_1 - \theta_2) P_{1/15}(\theta_1 - \theta_2) P_{2/5}(\theta_1 - \theta_2) P_{3/5}(\theta_1 - \theta_2)} \quad (S8)$$

$$F_{cc}^\sigma(\theta_1, \theta_2) = \{c_{33}^0 + c_{33}^1 \cosh(\theta_1 - \theta_2) + c_{33}^2 \cosh^2(\theta_1 - \theta_2) + c_{33}^3 \cosh^3(\theta_1 - \theta_2) + c_{33}^4 \cosh^4(\theta_1 - \theta_2)\} \cdot \left\{ -i \sinh\left(\frac{\theta_1 - \theta_2}{2}\right) \right\} T_{11/30}(\theta_1 - \theta_2) [T_{2/3}(\theta_1 - \theta_2)]^3 T_{7/15}(\theta_1 - \theta_2) [T_{2/5}(\theta_1 - \theta_2)]^3 T_{2/15}(\theta_1 - \theta_2) \cdot [T_{1/15}(\theta_1 - \theta_2)]^2 / \left\{ P_{11/30}(\theta_1 - \theta_2) P_{7/15}(\theta_1 - \theta_2) P_{2/15}(\theta_1 - \theta_2) [P_{2/3}(\theta_1 - \theta_2)]^2 P_{1/3}(\theta_1 - \theta_2) \cdot [P_{2/5}(\theta_1 - \theta_2)]^2 P_{3/5}(\theta_1 - \theta_2) P_{1/15}(\theta_1 - \theta_2) P_{14/15}(\theta_1 - \theta_2) \right\} \quad (S9)$$

$$F_{ab}^\sigma(\theta_1, \theta_2) = \{c_{12}^0 + c_{12}^1 \cosh(\theta_1 - \theta_2) + c_{12}^2 \cosh^2(\theta_1 - \theta_2)\} \frac{T_{4/5}(\theta_1 - \theta_2) T_{3/5}(\theta_1 - \theta_2) T_{7/15}(\theta_1 - \theta_2) T_{4/15}(\theta_1 - \theta_2)}{P_{4/5}(\theta_1 - \theta_2) P_{3/5}(\theta_1 - \theta_2) P_{7/15}(\theta_1 - \theta_2) P_{4/15}(\theta_1 - \theta_2)} \quad (S10)$$

$$F_{ac}^\sigma(\theta_1, \theta_2) = \{c_{13}^0 + c_{13}^1 \cosh(\theta_1 - \theta_2) + c_{13}^2 \cosh^2(\theta_1 - \theta_2) + c_{13}^3 \cosh^3(\theta_1 - \theta_2)\} \cdot \frac{T_{29/30}(\theta_1 - \theta_2) T_{7/10}(\theta_1 - \theta_2) T_{13/30}(\theta_1 - \theta_2) T_{1/10}(\theta_1 - \theta_2) [T_{11/30}(\theta_1 - \theta_2)]^2}{P_{29/30}(\theta_1 - \theta_2) P_{7/10}(\theta_1 - \theta_2) P_{13/30}(\theta_1 - \theta_2) P_{1/10}(\theta_1 - \theta_2) P_{11/30}(\theta_1 - \theta_2) P_{19/30}(\theta_1 - \theta_2)} \quad (S11)$$

$$F_{bc}^\sigma(\theta_1, \theta_2) = \{c_{23}^0 + c_{23}^1 \cosh(\theta_1 - \theta_2) + c_{23}^2 \cosh^2(\theta_1 - \theta_2) + c_{23}^3 \cosh^3(\theta_1 - \theta_2) + c_{23}^4 \cosh^4(\theta_1 - \theta_2)\} \cdot \frac{T_{25/30}(\theta_1 - \theta_2) T_{19/30}(\theta_1 - \theta_2) T_{9/30}(\theta_1 - \theta_2) [T_{7/30}(\theta_1 - \theta_2)]^2 [T_{13/30}(\theta_1 - \theta_2)]^2 T_{15/30}(\theta_1 - \theta_2)}{P_{25/30}(\theta_1 - \theta_2) P_{19/30}(\theta_1 - \theta_2) P_{9/30}(\theta_1 - \theta_2) P_{7/30}(\theta_1 - \theta_2) P_{23/30}(\theta_1 - \theta_2) P_{13/30}(\theta_1 - \theta_2) P_{17/30}(\theta_1 - \theta_2) P_{15/30}(\theta_1 - \theta_2)} \quad (S12)$$

where

$$T_\lambda(\theta) = \exp \left\{ 2 \int_0^\infty \frac{dt \cosh[(\lambda - 1/2)t]}{t \cosh(t/2) \sinh t} \sin^2 \frac{(i\pi - \theta)t}{2\pi} \right\} \quad \text{and} \quad P_\lambda(\theta) = \frac{\cos(\lambda\pi) - \cosh \theta}{2 \cos^2(\lambda\pi/2)}. \quad (S13)$$

The coefficients $\{c_{ij}^k\}$ in the above expressions are [15]:

$$\begin{aligned} c_{11}^0, c_{11}^1 &= -10.19307727, -2.09310293 \\ c_{22}^0, c_{22}^1, c_{22}^2, c_{22}^3 &= -500.2535896, -791.3745549, -338.8125724, -21.48559881 \\ c_{33}^0, c_{33}^1, c_{33}^2, c_{33}^3, c_{33}^4, c_{33}^5 &= -87821.70785, -267341.1276, -301093.9432, -150512.4122, -30166.99117, -1197.056497 \\ c_{12}^0, c_{12}^1, c_{12}^2 &= -70.29218939519, -71.792063506, -7.9790221816 \\ c_{13}^0, c_{13}^1, c_{13}^2, c_{13}^3 &= -7049.622303, -13406.48877, -6944.416956, -582.2557366 \\ c_{23}^0, c_{23}^1, c_{23}^2, c_{23}^3, c_{23}^4 &= -3579.556465, -8436.850081, -6618.297073, -1846.579035, -92.73452314 \end{aligned}$$

The relevant three-particle form factor of the E_8 model is,

$$F_{aaa}^\sigma(\theta_1, \theta_2, \theta_3) = Q_{aaa}^\sigma(\theta_1, \theta_2, \theta_3) \frac{F_{aa}^{\min}(\theta_1 - \theta_2)}{(e^{\theta_1} + e^{\theta_2}) D_{aa}(\theta_1 - \theta_2)} \frac{F_{aa}^{\min}(\theta_1 - \theta_3)}{(e^{\theta_1} + e^{\theta_3}) D_{aa}(\theta_1 - \theta_3)} \frac{F_{aa}^{\min}(\theta_2 - \theta_3)}{(e^{\theta_2} + e^{\theta_3}) D_{aa}(\theta_2 - \theta_3)} \quad (S14)$$

where[14, 15]

$$\frac{F_{aa}^{\min}(\theta_i - \theta_j)}{D_{aa}(\theta_i - \theta_j)} = \left\{ -i \sinh \left(\frac{\theta_i - \theta_j}{2} \right) \right\} \frac{T_{2/3}(\theta_i - \theta_j) T_{2/5}(\theta_i - \theta_j) T_{1/15}(\theta_i - \theta_j)}{P_{2/3}(\theta_i - \theta_j) P_{2/5}(\theta_i - \theta_j) P_{1/15}(\theta_i - \theta_j)} \quad (S15)$$

and[16]

$$\begin{aligned} Q_{aaa}^\sigma(\theta_1, \theta_2, \theta_3) &= 1148.690509e^{3\theta_1} + 46.76252978e^{4\theta_1 - \theta_2} + 1148.690509e^{3\theta_2} + 3703.911733e^{2\theta_1 + \theta_2} \\ &+ 3703.911733e^{\theta_1 + 2\theta_2} + 46.76252978e^{-\theta_1 + 4\theta_2} + 4.354182251e^{4\theta_1 + \theta_2 - 2\theta_3} + 46.76252978e^{3\theta_1 + 2\theta_2 - 2\theta_3} \\ &+ 46.76252978e^{2\theta_1 + 3\theta_2 - 2\theta_3} + 4.354182251e^{\theta_1 + 4\theta_2 - 2\theta_3} + 46.76252978e^{4\theta_1 - \theta_3} + 604.2577928e^{3\theta_1 + \theta_2 - \theta_3} \\ &+ 1148.690509e^{2\theta_1 + 2\theta_2 - \theta_3} + 604.2577928e^{\theta_1 + 3\theta_2 - \theta_3} + 46.76252978e^{4\theta_2 - \theta_3} + 1148.690509e^{3\theta_3} \\ &+ 3703.911733e^{2\theta_1 + \theta_3} + 4.354182251e^{4\theta_1 - 2\theta_2 + \theta_3} + 604.2577928e^{3\theta_1 - \theta_2 + \theta_3} + 6286.815608e^{\theta_1 + \theta_2 + \theta_3} \\ &+ 3703.911733e^{2\theta_2 + \theta_3} + 604.2577928e^{-\theta_1 + 3\theta_2 + \theta_3} + 4.354182251e^{-2\theta_1 + 4\theta_2 + \theta_3} + 3703.911733e^{\theta_1 + 2\theta_3} \\ &+ 46.76252978e^{3\theta_1 - 2\theta_2 + 2\theta_3} + 1148.690509e^{2\theta_1 - \theta_2 + 2\theta_3} + 3703.911733e^{\theta_2 + 2\theta_3} + 1148.690509e^{-\theta_1 + 2\theta_2 + 2\theta_3} \\ &+ 46.76252978e^{-2\theta_1 + 3\theta_2 + 2\theta_3} + 46.76252978e^{2\theta_1 - 2\theta_2 + 3\theta_3} + 604.2577928e^{\theta_1 - \theta_2 + 3\theta_3} + 604.2577928e^{-\theta_1 + \theta_2 + 3\theta_3} \\ &+ 46.76252978e^{-2\theta_1 + 2\theta_2 + 3\theta_3} + 46.76252978e^{-\theta_1 + 4\theta_3} + 4.354182251e^{\theta_1 - 2\theta_2 + 4\theta_3} + 46.76252978e^{-\theta_2 + 4\theta_3} \\ &+ 4.354182251e^{-2\theta_1 + \theta_2 + 4\theta_3} \end{aligned} \quad (S16)$$

Derivation of $S_{1,1}(\omega, q)$ (Eq.(19) of the Main Text)

We calculate $S_{1,1}$ using the finite volume regularization scheme [18, 19]. We have $D_{00} = \langle \sigma \rangle_0^2$, and

$$\begin{aligned} D_{11}(x, t) &= \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta'_1}{2\pi} F_2^\sigma(\theta_1 + i\pi, \theta'_1) F_2^\sigma(\theta'_1 + i\pi, \theta_1) \cdot \\ &\cdot e^{-\beta \Delta_1 \cosh \theta_1} e^{-i(\Delta_1 \sinh \theta_1 - \Delta_2 \sinh \theta'_1)x} e^{-i(\Delta_2 \cosh \theta'_1 - \Delta_1 \cosh \theta_1)t} + 2 \langle \sigma \rangle_0 F_{2s}^\sigma \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} e^{-\beta \Delta_1 \cosh \theta_1} \end{aligned} \quad (S17)$$

From now until the calculation of D_{22} , we will focus on the time-dependent parts, i.e., the connected pieces of the correlation functions. The time-independent parts, i.e., the disconnected pieces, will be discussed after the analysis on the time-dependent parts of D_{22} . We then have

$$S_{11}(q, \omega) = \int d\theta_1 d\theta'_1 F_2^\sigma(\theta_1 + i\pi, \theta'_1) F_2^\sigma(\theta'_1 + i\pi, \theta_1) e^{-\beta \Delta_1 \cosh \theta_1} \cdot \delta(q + \Delta_1 \sinh \theta_1 - \Delta_2 \sinh \theta'_1) \delta(\omega + \Delta_1 \cosh \theta_1 - \Delta_2 \cosh \theta'_1) \quad (\text{S18})$$

Denote

$$\begin{cases} y = \Delta_1 \sinh \theta_1 - \Delta_2 \sinh \theta'_1 \\ z = \Delta_1 \cosh \theta_1 - \Delta_2 \cosh \theta'_1 \end{cases} \quad (\text{S19})$$

then

$$d\theta_1 d\theta'_1 = \left| \frac{\partial(\theta_1, \theta'_1)}{\partial(y, z)} \right| dy dz = \frac{dy dz}{\Delta_1 \Delta_2 \sqrt{\cosh^2(\theta_1 - \theta'_1) - 1}} \quad (\text{S20})$$

Noticing that the integration ranges for new variables y and z run from $-\infty$ to $+\infty$, we can easily perform the integral in the structure factor and find

$$S_{11}(q, \omega) = \int d\theta_1 d\theta'_1 F_2^\sigma(\theta_1 + i\pi, \theta'_1) F_2^\sigma(\theta'_1 + i\pi, \theta_1) \delta(q + \Delta_1 \sinh \theta_1 - \Delta_2 \sinh \theta'_1) \cdot \delta(\omega + \Delta_1 \cosh \theta_1 - \Delta_2 \cosh \theta'_1) e^{-\beta \Delta_1 \cosh \theta_1} \quad (\text{S21})$$

$$= \frac{F_2^\sigma(\theta_1 + i\pi, \theta'_1) F_2^\sigma(\theta'_1 + i\pi, \theta_1) e^{-\beta \Delta_2 \cosh \theta'_1} e^{\beta \omega}}{\Delta_1 \Delta_2 \sqrt{\cosh^2(\theta_1 - \theta'_1) - 1}} \Big|_{\theta_1 - \theta'_1 = \alpha} + \frac{F_2^\sigma(\theta_1 + i\pi, \theta'_1) F_2^\sigma(\theta'_1 + i\pi, \theta_1) e^{-\beta \Delta_2 \cosh \theta'_1} e^{\beta \omega}}{\Delta_1 \Delta_2 \sqrt{\cosh^2(\theta_1 - \theta'_1) - 1}} \Big|_{\theta_1 - \theta'_1 = -\alpha} \quad (\text{S22})$$

$$= \frac{|F_2^\sigma(\alpha + i\pi, 0)|^2 e^{-\beta \Delta_1 \cosh \theta_{1+}}}{\Delta_1 \Delta_2 \sqrt{\cosh^2 \alpha - 1}} + \frac{|F_2^\sigma(-\alpha + i\pi, 0)|^2 e^{-\beta \Delta_1 \cosh \theta_{1-}}}{\Delta_1 \Delta_2 \sqrt{\cosh^2 \alpha - 1}} \quad (\text{S23})$$

$$= \frac{|F_2^\sigma(\alpha + i\pi, 0)|^2 (e^{-\beta \Delta_1 \cosh \theta_{1+}} + e^{-\beta \Delta_1 \cosh \theta_{1-}})}{\Delta_1 \Delta_2 \sqrt{\cosh^2 \alpha - 1}} \quad (\text{S24})$$

where

$$\Gamma \equiv \cosh \alpha = \cosh(\theta_1 - \theta'_1) = \frac{\Delta_1^2 + \Delta_2^2 - (\omega^2 - q^2)}{2\Delta_1 \Delta_2} \quad (\text{S25})$$

$$\cosh \theta_{1\pm} = \frac{\omega \Delta_1 - \omega \Delta_2 \Gamma \pm q \Delta_2 \sqrt{\Gamma^2 - 1}}{q^2 - \omega^2} \quad (\text{S26})$$

In going from Eq. (S22) to Eq. (S23), we have used the fact that the form factor is only dependent on the difference between any two rapidities. We then recover Eq.(19) of the main text.

Calculation of $S_{1,1}(\omega, q)$ (Eq.(21) of the Main Text) for Equal Masses at Low Frequencies

In Eq. (S21) above, the q integration followed by the θ'_1 integration gives rise to

$$S_{11}(\omega) = \frac{2}{\Delta_i} \int d\theta_1 \left\{ \left[|F_2^\sigma(\theta_1 - \theta'_1(\theta_1, \omega) + i\pi, 0)|^2 e^{-\beta \Delta_i \cosh \theta_1} \right] / \sqrt{\left(\cosh \theta_1 + \frac{\omega}{\Delta_i} \right)^2 - 1} \right\} \quad (\text{S27})$$

We make a further variable transform $\cosh \theta_1 = x - \omega_i$ ($\omega_i = \omega/(2\Delta_i)$) and have

$$S_{1,1}(\omega_i)|_{\Delta_1=\Delta_2=\Delta_i} = \frac{2e^{\omega/(2T)}}{\Delta} \int_{1+\omega_i}^{\infty} dx \frac{(F(\omega_i, q_-(x)) + F(\omega_i, q_+(x))) \exp\{-\frac{\Delta_i}{T} x\}}{\sqrt{(x+1+\omega_i)(x+1-\omega_i)(x-1+\omega_i)(x-1-\omega_i)}} \quad (\text{S28})$$

where $\omega_i = \omega/(2\Delta_i)$, and

$$F(\omega_i, q_{\pm}(x)) = \left| F_2^{\sigma}(\operatorname{arccosh} \left[1 + \left(-\omega_i^2 + x^2 - 1 \pm \sqrt{(x-1-\omega_i)(x-1+\omega_i)(x+1+\omega_i)(x+1-\omega_i)} \right) \right] + i\pi, 0) \right| \quad (\text{S29})$$

We can also get Eq. (S28) by making variable transform $x = -\omega/(2\Delta_i) + [q/(2\Delta_i)] \sqrt{(q^2 - \omega^2 + 4\Delta_i^2)/(q^2 - \omega^2)}$ for the q integration over Eq. (S24). The exponential-decaying factor in the integrand of Eq. (S28) indicates that the dominant contribution come from the regime where x is close to $1 + \omega_i$. Since ω_i is small, in this regime we can approximate $F(\omega_i, q_{\pm}(x))$ as

$$F(\omega_i, q_{\pm}(x)) \approx |F_2^{\sigma}(i\pi, 0)|_{\Delta_1=\Delta_2=\Delta_i}^2 (i = a, b, c, d, e, f, g, h) \quad (\text{S30})$$

Then we have

$$S_{1,1}(\omega \rightarrow 0, \omega/\Delta_a \ll 1)|_{\Delta_1=\Delta_2=\Delta_i} \quad (\text{S31})$$

$$\approx \frac{4e^{\omega/(2T)}}{\Delta} \int_{1+\omega_i}^{\infty} dx \frac{|F_{ii}(i\pi, 0)|^2 \exp\{-\frac{\Delta_i}{T}x\}}{\sqrt{(x+1+\omega_i)(x+1-\omega_i)(x-1+\omega_i)(x-1-\omega_i)}} \quad (\text{S32})$$

$$\approx \begin{cases} \frac{2e^{\omega/(2T)}}{\Delta_i} \frac{|F_{ii}(i\pi, 0)|^2}{\Delta_i} e^{-\Delta_i/T} \left\{ -\ln \frac{\omega}{4T} - \gamma_E + \dots \right\} & (\omega \ll T \ll \Delta_i) \\ \frac{2e^{-\Delta_i/T}}{\Delta_i} \frac{|F_{ii}(i\pi)|^2}{\Delta_i} \left\{ \sqrt{\frac{\pi T}{\omega}} - \frac{\sqrt{\pi}}{4} \left(\frac{T}{\omega}\right)^{3/2} + \dots \right\} & (T \ll \omega \ll \Delta_i) \end{cases} \quad (\text{S33})$$

where $|F_2^{\sigma}(i\pi, 0)|_{\Delta_1=\Delta_2=\Delta_i}^2 \approx 65$.

Derivation of $S_{1,2}(\omega, q)$ (Eq.(22) of the Main Text)

We again use the finite volume regularization scheme [18, 19], and have

$$D_{12}(x, t) = C_{12} - Z_1 C_{01} \quad (\text{S34})$$

$$= \frac{1}{2} \int_{C_+} \frac{d\theta_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta'_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta'_2}{2\pi} F_3^{\sigma}(\theta_1 + i\pi, \theta'_1, \theta'_2) F_3^{\sigma}(\theta_1 + i\pi, \theta'_2, \theta'_1) \quad (\text{S35})$$

$$e^{-\beta\Delta_a \cosh \theta_1} e^{-ix\Delta_a (\sinh \theta_1 - \sinh \theta'_1 - \sinh \theta'_2)} e^{-it\Delta_a (\cosh \theta'_1 + \cosh \theta'_2 - \cosh \theta_1)} \quad (\text{S36})$$

$$+ \int_{-\infty}^{\infty} \frac{d\theta'_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta'_2}{2\pi} \left\{ e^{-\beta\Delta_a \cosh \theta'_1} e^{i\Delta_a x \sinh \theta'_2} e^{-it\Delta_a \cosh \theta'_2} S(\theta'_2 - \theta'_1) F_1^{\sigma} F_{3c}^{\sigma}(\theta'_1 | \theta'_1 \theta'_2) + (\theta'_1 \leftrightarrow \theta'_2) \right\} \quad (\text{S37})$$

$$- (F_1^{\sigma})^2 \int_{-\infty}^{\infty} \frac{d\theta'_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta'_2}{2\pi} e^{-\beta\Delta_a \cosh \theta'_1} e^{i\Delta_a x \sinh \theta'_2} e^{-it\Delta_a \cosh \theta'_2}. \quad (\text{S38})$$

$$(\Delta_a x \cosh \theta'_1 + \Delta_a (i\beta + t) \sinh \theta'_1) [S(\theta'_1 - \theta'_2) - 1] \quad (\text{S39})$$

$$- (F_1^{\sigma})^2 \int_{-\infty}^{\infty} \frac{d\theta'_2}{2\pi} e^{-\beta\Delta_a \cosh \theta'_2} e^{i\Delta_a x \sinh \theta'_2} e^{-it\Delta_a \cosh \theta'_2} \quad (\text{S40})$$

where C_+ is used to denote the integration contour from $-\infty$ to ∞ slightly above the real axis on the rapidity complex plane, and [18, 19]

$$F_3^{\sigma}(\theta_1 + i\pi, \theta'_1, \theta'_2) = \frac{i(1 - S(\theta'_1 - \theta'_2)) F_1^{\sigma}}{\theta_1 - \theta'_1} + \frac{i(S(\theta'_1 - \theta'_2) - 1) F_1^{\sigma}}{\theta_1 - \theta'_2} + F_{3rc}^{\sigma}(\theta_1 + i\pi | \theta'_1, \theta'_2) \quad (\text{S41})$$

where S_{aa} is the scattering matrix for $a - a$ channel, and $F_{3rc}^{\sigma}(\theta_1 + i\pi | \theta'_1, \theta'_2)$ is regular on real axis.

For $x = 0$, it's easy to see that the last three terms do not contribute to low-frequency ($\omega \ll \Delta_a$) response of local DSF. From the first integration we have

$$S_{12}(q, \omega) = \frac{1}{2} \int \frac{d\theta_1}{2\pi} \int_{-\infty}^{\infty} d\theta'_1 \int_{-\infty}^{\infty} d\theta'_2 F_3^{\sigma}(\theta_1 + i\pi + i\varepsilon, \theta'_1, \theta'_2) F_3^{\sigma}(\theta_1 + i\pi + i\varepsilon, \theta'_2, \theta'_1) \quad (\text{S42})$$

$$\delta(q + \Delta_a \sinh \theta_1 - \Delta_a (\sinh \theta'_1 + \sinh \theta'_2)) \delta(\omega + \Delta_a \cosh \theta_1 - \Delta_a (\cosh \theta'_1 + \cosh \theta'_2)) \quad (\text{S43})$$

The energy-momentum conservation yields

$$\begin{cases} 0 = q + \sinh \theta_1 - \sinh \theta'_1 - \sinh \theta'_2 \\ 0 = \omega + \cosh \theta_1 - \cosh \theta'_1 - \cosh \theta'_2 \end{cases} \quad (\text{S44})$$

For F_{3rc}^σ we can integrate over θ'_1 and θ'_2 , yielding (because the masses of three particles are equal to each other, $S_{21}(q, \omega) = S_{12}(q, \omega)$)

$$S_{(12)+(21)}(q, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_1}{\sqrt{(f(\tilde{\omega}, \tilde{q}, \theta_1) - 1)^2 - 1}} F_{3rc}^\sigma(\theta_1 + i\pi | \ln z_+, \ln z_-) F_{3rc}^\sigma(\theta_1 + i\pi | \ln z_-, \ln z_+) e^{-\beta \Delta_a \cosh \theta_1} \quad (S45)$$

with

$$z_{\pm} = \frac{1}{2}(\tilde{\omega} + \cosh \theta + \tilde{q} + \sinh \theta) \left(1 \pm 2\sqrt{1 - 2/f(\tilde{\omega}, \tilde{q}, \theta)} \right), \quad (S46)$$

and

$$f(\tilde{\omega}, \tilde{q}, \theta) = \left[(\tilde{\omega} + \cosh \theta)^2 - (\tilde{q} + \sinh \theta)^2 \right] / 2 \quad (S47)$$

where $\tilde{\omega} = \omega/\Delta_a$ and $\tilde{q} = q/\Delta_a$. Thus we recover Eq. (22) of the main text. The energy-momentum conservation gives a constraint: $f(\tilde{\omega}, \tilde{q}, \theta) \geq 2$, i.e., $(\tilde{\omega} - \tilde{q})e^{2\theta} + (\tilde{\omega}^2 - \tilde{q}^2 - 3)e^\theta + \tilde{\omega} + \tilde{q} \geq 0$. This constraint allows zero in the denominator of the integrand in Eq. (S45), which is a branch point. This can be clearly shown after a variable transform $e^\theta \rightarrow x$ and expanding x around zero. Thus the integration will smooth out the superficial singularity leaving us with a regular integration over θ . Furthermore, if $\tilde{\omega} > \tilde{q} \geq 0$, we can get the constraint for rapidity $0 < 2e^\theta < -\frac{\tilde{\omega}^2 - \tilde{q}^2 - 3}{\tilde{\omega} - \tilde{q}} - \sqrt{\left(\frac{\tilde{\omega}^2 - \tilde{q}^2 - 3}{\tilde{\omega} - \tilde{q}}\right)^2 - 4\frac{\tilde{\omega} + \tilde{q}}{\tilde{\omega} - \tilde{q}}}$ or $2e^\theta > -\frac{\tilde{\omega}^2 - \tilde{q}^2 - 3}{\tilde{\omega} - \tilde{q}} + \sqrt{\left(\frac{\tilde{\omega}^2 - \tilde{q}^2 - 3}{\tilde{\omega} - \tilde{q}}\right)^2 - 4\frac{\tilde{\omega} + \tilde{q}}{\tilde{\omega} - \tilde{q}}}$. However, it's easy to see in these two ranges that, because $\tilde{\omega} \ll 1$, we will have $\cosh \theta \sim 1/\tilde{\omega}$, making it negligible in the zero frequency limit. If $\tilde{\omega} \leq \tilde{q}$, we get constraint on the rapidity of θ as (without loss of generality we choose $\tilde{\omega} > 0$): $0 < 2e^\theta < \left[(\tilde{\omega}^2 - \tilde{q}^2 - 3) + \sqrt{(\tilde{\omega}^2 - \tilde{q}^2 - 1)(\tilde{\omega}^2 - \tilde{q}^2 - 9)} \right] / (\tilde{q} - \tilde{\omega}) \equiv \mu(\tilde{\omega}, \tilde{q})$. We can then determine the maximum of $\mu(\tilde{\omega}, \tilde{q})$ to be located at $\tilde{q}_m = \sqrt{(3 - \tilde{\omega})(1 - \tilde{\omega})}$. Again recalling $\tilde{\omega} \ll 1$, we have $\cosh \theta \gtrsim 2 - \tilde{\omega}$. This indicates that a small region of \tilde{q} exists, in which $\cosh \theta$ is slightly smaller than 2. Therefore, we will include in our numerical calculation the channels $D_{1,2} + D_{2,1}$.

For the leftover two parts in Eq. (S41), we have

$$\frac{i(1 - S(\theta'_1 - \theta'_2)) F_1^\sigma}{\theta_1 - \theta'_1 + i\varepsilon} = P \frac{i(1 - S(\theta'_1 - \theta'_2)) F_1^\sigma}{\theta_1 - \theta'_1} - i\pi\delta(\theta_1 - \theta'_1), \quad (S48)$$

$$\frac{i(S(\theta'_1 - \theta'_2) - 1) F_1^\sigma}{\theta_1 - \theta'_2 + i\varepsilon} = P \frac{i(S(\theta'_1 - \theta'_2) - 1) F_1^\sigma}{\theta_1 - \theta'_2} - i\pi\delta(\theta_1 - \theta'_2). \quad (S49)$$

Here P denotes principal value integration. The parts of $F_3^\sigma(\theta_1 + i\pi + i\varepsilon, \theta'_1, \theta'_2) F_3^\sigma(\theta_1 + i\pi + i\varepsilon, \theta'_2, \theta'_1)$ do not contribute: after integrating over θ'_1 or θ'_2 , which leaves us with $e^{-it\Delta_a \cosh \theta_1}$; since $\omega \ll \Delta_a$, it vanishes for the local low-frequency dynamics. For the parts of $F_3^\sigma(\theta_1 + i\pi + i\varepsilon, \theta'_1, \theta'_2) F_3^\sigma(\theta_1 + i\pi + i\varepsilon, \theta'_2, \theta'_1)$ containing $1/(\theta_1 - \theta'_{1/2} + i\varepsilon)^2$, we can finish an integration by part, which leaves us with only a simple principal-value integration. We can repeat the discussions for the part having delta function, and show that it does not have any contribution. Consider now all the leftover parts in $F_3^\sigma(\theta_1 + i\pi + i\varepsilon, \theta'_1, \theta'_2) F_3^\sigma(\theta_1 + i\pi + i\varepsilon, \theta'_2, \theta'_1)$. Since they are all principal-value-type integration, they do not encounter any singularity. Following the discussion on the integration containing integrand of $F_{3rc}^\sigma(\theta_1 + i\pi | \ln x_+, \ln x_-) F_{3rc}^\sigma(\theta_1 + i\pi | \ln x_-, \ln x_+)$, they will have similar contributions as those for Eq. (S45). They will likewise be included in our numerical calculations.

Calculation of D_{22}

Using the finite volume regularization scheme [18, 19], we have

$$D_{aa,aa}(x, t) = C_{22} - Z_1 C_{11} + (Z_1^2 - Z_2) C_{00} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 \quad (S50)$$

We analyze the integrals in $D_{aa,aa}$ one by one. In all the following analyses, we focus on the low-frequency regime. (High-frequency regime is relatively straightforward, where the steepest descent method can be applied directly.) The

three integrals I_1 , I_2 and I_3 are time-independent,

$$I_1 = -2 \int \frac{d\theta_1}{2\pi} F_2^\sigma(i\pi, 0) \langle \sigma \rangle e^{-2\beta\Delta_a \cosh \theta_1} \quad (\text{S51})$$

$$I_2 = \frac{1}{2} \int \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} (F_2^\sigma(i\pi, 0))^2 e^{-\beta\Delta_a (\cosh \theta_1 + \cosh \theta_2)} \quad (\text{S52})$$

$$I_3 = \int \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} F_{4s}^\sigma(\theta_1, \theta_2) \langle \sigma \rangle e^{-\beta\Delta_a (\cosh \theta_1 + \cosh \theta_2)} \quad (\text{S53})$$

where

$$\begin{aligned} F_{4s}^\sigma(\theta_1, \theta_2) &= \lim_{\varepsilon \rightarrow 0} F_4^\sigma(\theta_1 + i\pi + \varepsilon, \theta_2 + i\pi + \varepsilon, \theta_2, \theta_1) \\ &= \frac{2iF_2^\sigma(\theta_2 + i\pi, \theta_1) [S_{aa}(\theta_1 - \theta_2) - S_{aa}(\theta_2 - \theta_1)]}{\theta_1 - \theta_2} + F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi | \theta_2, \theta_1) \end{aligned} \quad (\text{S54})$$

Since $\lim_{z \rightarrow 0} [(S_{aa}(z) - S_{aa}(-z))/z] = 2S'_{aa}(0)$ is finite, and F_{4rc}^σ is a regular function on real axis [18, 19], the whole integrand in I_3 is regular. As we mentioned before we will return to the discussion of these constant parts.

The integral I_4 is

$$I_4 = - \int \int \frac{d\theta_1 d\theta'_1}{(2\pi)^2} (F_2^\sigma(\theta_1 + i\pi, \theta'_1))^2 e^{-2\beta\Delta_a \cosh \theta_1} e^{-ix\Delta_a (\sinh \theta_1 - \sinh \theta'_1)} e^{-it\Delta_a (\cosh \theta'_1 - \cosh \theta_1)} \quad (\text{S55})$$

I_4 has the same integral structure as seen in the calculation of S_{11} , except for a different thermal weight-factor. So we will have a similar $\ln(\omega/T)$ divergence in the low-frequency regime as in S_{11} . However, it is associated with a $e^{-2\Delta_a/T}$ factor, and thus negligible compared with S_{11} .

The integral I_5 is

$$I_5 = - \int \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} (F_2^\sigma(\theta_2 + i\pi, \theta_1))^2 e^{-\beta\Delta_a (\cosh \theta_1 + \cosh \theta_2)} e^{-ix\Delta_a (\sinh \theta_2 - \sinh \theta_1)} e^{-it\Delta_a (\cosh \theta_1 - \cosh \theta_2)} \quad (\text{S56})$$

Again we have a similar integrand structure as in S_{11} , and therefore the divergence in the low-frequency regime in I_5 will not be stronger than $\ln(\omega/T)$; the thermal factor $e^{-2\Delta_a/T}$ again makes it negligible compared with S_{11} .

The integral I_6 is $I_6 = I_6^{(1)} + I_6^{(2)}$, with

$$\begin{aligned} I_6^{(1)} &= \int \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \int \frac{d\theta'_1}{2\pi} (F_2^\sigma(\theta_1 + i\pi, \theta'_1))^2 [(1 - S(\theta'_1 - \theta_1)S(\theta_1 - \theta_2)) (-\Delta_a x \cosh \theta_1 + \Delta_a t \sinh \theta_1)] \\ &\quad e^{-\beta\Delta_a (\cosh \theta_1 + \cosh \theta_2)} e^{-ix\Delta_a (\sinh \theta_2 - \sinh \theta'_1)} e^{-it\Delta_a (\cosh \theta'_1 - \cosh \theta_2)} \end{aligned} \quad (\text{S57})$$

$$\begin{aligned} I_6^{(2)} &= - \int \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \int \frac{d\theta'_1}{2\pi} (F_2^\sigma(\theta_1 + i\pi, \theta'_1))^2 \varphi(\theta'_1 - \theta_1) S(\theta'_1 - \theta_1) S(\theta_1 - \theta_2) \\ &\quad e^{-\beta\Delta_a (\cosh \theta_1 + \cosh \theta_2)} e^{-ix\Delta_a (\sinh \theta_2 - \sinh \theta'_1)} e^{-it\Delta_a (\cosh \theta'_1 - \cosh \theta_2)} \end{aligned} \quad (\text{S58})$$

where

$$\varphi(\theta'_1 - \theta_1) S(\theta'_1 - \theta_1) S(\theta_1 - \theta_2) = -\frac{i}{S(\theta'_1 - \theta_1)} \frac{dS(\theta'_1 - \theta_1)}{d\theta'_1} S(\theta'_1 - \theta_1) S(\theta_1 - \theta_2) = -iS(\theta_1 - \theta_2) \frac{dS(\theta'_1 - \theta_1)}{d\theta'_1} \quad (\text{S59})$$

We will see that combining $I_6^{(1)}$ and part of I_8 gives zero contribution to the local dynamics. So we consider $I_6^{(2)}$. Since the time-space oscillation factor in $I_6^{(2)}$, $e^{-ix\Delta_a (\sinh \theta_2 - \sinh \theta'_1)} e^{-it\Delta_a (\cosh \theta'_1 - \cosh \theta_2)}$, is independent of rapidity θ_1 , and the leftover integrand is regular on the real axis, we can apply the steepest descent method for θ_1 with saddle point at $\theta_1 = 0$,

$$I_6^{(2)} \sim \sqrt{\frac{T}{\Delta_a}} e^{-\Delta_a/T} \int \int \frac{d\theta_1 d\theta'_1}{(2\pi)^2} (F_2^\sigma(i\pi, \theta'_1))^2 S(-\theta_2) \left[i \frac{dS(\theta'_1)}{d\theta'_1} \right] e^{-\beta\Delta_a \cosh \theta_2} e^{-ix\Delta_a (\sinh \theta_2 - \sinh \theta'_1)} e^{-it\Delta_a (\cosh \theta'_1 - \cosh \theta_2)} \quad (\text{S60})$$

The leftover integral has similar structure as in S_{11} and, in low-frequency regime,

$$\left| I_6^{(2)}(\omega) \right| \sim \left| \sqrt{\frac{T}{\Delta_a}} \frac{e^{-2\Delta_a/T} e^{\omega/2T} |F_2^\sigma(i\pi, 0)|^2}{\Delta_a} S(0) S'(0) \ln \frac{\omega}{4T} \right| \quad (\omega \ll T \ll \Delta_a). \quad (\text{S61})$$

Thus, its contribution to the low-energy local dynamics is negligible compared with S_{11} .

The integral I_7 is

$$I_7 = 2 \int \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} P \int \frac{d\theta'_1}{2\pi} F_{4ss}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi | \theta'_1, \theta_1) F_2^\sigma(\theta_2 + i\pi, \theta'_1) e^{-\beta\Delta_a(\cosh\theta_1 + \cosh\theta_2)} e^{-ix\Delta_a(\sinh\theta_2 - \sinh\theta'_1)} e^{-it\Delta_a(\cosh\theta'_1 - \cosh\theta_2)} \quad (S62)$$

where

$$F_{4ss}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi | \theta'_1, \theta_1) = \frac{i(S_{aa}(\theta_1 - \theta_2) + 1)}{\theta_1 - \theta'_1} F_2^\sigma(\theta_2 + i\pi, \theta'_1) + F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi | \theta'_1, \theta_1) \quad (S63)$$

Consider the part containing $F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi | \theta'_1, \theta_1)$. Since $F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi | \theta'_1, \theta_1)$ is not singular on the real axis, this part of the integration behaves similarly as that in $I_6^{(2)}$, making its contribution negligible in the low-frequency regime. Consider next the part containing $\frac{i(S_{aa}(\theta_1 - \theta_2) + 1)}{\theta_1 - \theta'_1} F_2^\sigma(\theta_2 + i\pi, \theta'_1)$. The integration here is still well defined in the sense of principal-value integration over θ'_1 . Recall the definition of principal-value integration:

$$P \int_L \frac{h(\tau)}{\tau - t} d\tau = \int_L \frac{h(\tau) - h(t)}{\tau - t} d\tau + h(t) \ln \frac{b - t}{t - a} \quad (S64)$$

where t lies on curve L (not at end points). Since θ'_1 integration is on real axis,

$$P \int_R \frac{h(\theta'_1)}{\theta_1 - \theta'_1} d\theta'_1 = P \int_{-\infty}^{\infty} \frac{h(\theta'_1)}{\theta_1 - \theta'_1} d\theta'_1 = - \int_L \frac{h(\theta'_1) - h(\theta_1)}{\theta'_1 - \theta_1} d\tau \quad (S65)$$

In our case,

$$h(\theta'_1) = i(S_{aa}(\theta_1 - \theta_2) + 1) [F_2^\sigma(\theta_2 + i\pi, \theta'_1)]^2 e^{i\Delta_a x \sinh\theta'_1} e^{-it\Delta_a \cosh\theta'_1} \quad (S66)$$

The function associated with $e^{-\beta\Delta_a \cosh\theta_1}$ is not singular on real axis. Thus we can apply steepest decent method for θ_1 , leaving us θ'_1 and θ_2 integrations as

$$I_7 \sim 2 \sqrt{\frac{T}{\Delta_a}} e^{-\beta\Delta_a} \iint_R \frac{d\theta_2 d\theta'_1}{(2\pi)^2} \frac{i[S_{aa}(-\theta_2) + 1] \left[(F_2^\sigma(\theta_2 + i\pi, \theta'_1))^2 - (F_2^\sigma(\theta_2 + i\pi, 0))^2 \right]}{\theta'_1} e^{-\beta \cosh\theta_2} e^{-i\Delta_a x(\sinh\theta_2 - \sinh\theta'_1)} e^{-it\Delta_a(\cosh\theta'_1 - \cosh\theta_2)} \quad (S67)$$

For the leftover integration, we encounter a structure similar as in S_{11} . Therefore the part involving principal-value integraion gives contribution at the order of $\frac{1}{\Delta_a} \sqrt{\frac{T}{\Delta_a}} e^{-2\Delta_a/T} |F_2^\sigma(i\pi, 0)|^2 \ln \frac{\omega}{4T}$. Combining with the other part's contribution we have

$$I_7(\omega) \sim \frac{1}{\Delta_a} \sqrt{\frac{T}{\Delta_a}} e^{-2\Delta_a/T} |F_2^\sigma(i\pi, 0)|^2 \ln \frac{\omega}{4T} \quad (\omega \ll T \ll \Delta_a) \quad (S68)$$

We conclude that I_7 's contribution to low-energy local dynamics is negligible compared with S_{11} .

The integral I_8 is

$$I_8 = \frac{1}{4} \int \int \int \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} F_4^\sigma(\theta_2 + i\pi, \theta_1 + i\pi, \theta'_1, \theta'_2) F_4^\sigma(\theta_1 + i\pi, \theta_2 + i\pi, \theta'_2, \theta'_1) e^{-\beta\Delta_a(\cosh\theta_1 + \cosh\theta_2)} e^{-i\Delta_a x(\sinh\theta_1 + \sinh\theta_2 - \sinh\theta'_1 - \sinh\theta'_2)} e^{-it\Delta_a(\cosh\theta'_1 + \cosh\theta'_2 - \cosh\theta_1 - \cosh\theta_2)} \quad (S69)$$

where

$$F_4^\sigma(\theta_2 + i\pi, \theta_1 + i\pi, \theta'_1, \theta'_2) F_4^\sigma(\theta_1 + i\pi, \theta_2 + i\pi, \theta'_2, \theta'_1) \quad (S70)$$

$$= F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1 + i\varepsilon, \theta'_2 + i\varepsilon) F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi|\theta'_2 + i\varepsilon, \theta'_1 + i\varepsilon) \quad (S71)$$

$$+ F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1 + i\varepsilon, \theta'_2 + i\varepsilon) \left[\frac{E}{\theta_1 - \theta'_2 - i\varepsilon} + \frac{F}{\theta_1 - \theta'_1 - i\varepsilon} + \frac{G}{\theta_2 - \theta'_2 - i\varepsilon} + \frac{H}{\theta_2 - \theta'_1 - i\varepsilon} \right] \quad (S72)$$

$$+ F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi|\theta'_2 + i\varepsilon, \theta'_1 + i\varepsilon) \left[\frac{A}{\theta_2 - \theta'_1 - i\varepsilon} + \frac{B}{\theta_2 - \theta'_2 - i\varepsilon} + \frac{C}{\theta_1 - \theta'_1 - i\varepsilon} + \frac{D}{\theta_1 - \theta'_2 - i\varepsilon} \right] \quad (S73)$$

$$+ \frac{AH}{(\theta_2 - \theta'_1 - i\varepsilon)^2} + \frac{BG}{(\theta_2 - \theta'_2 - i\varepsilon)^2} + \frac{CF}{(\theta_1 - \theta'_1 - i\varepsilon)^2} + \frac{DE}{(\theta_1 - \theta'_2 - i\varepsilon)^2} + \frac{AE + DH}{(\theta_2 - \theta'_1 - i\varepsilon)(\theta_1 - \theta'_2 - i\varepsilon)} \quad (S74)$$

$$+ \frac{AF + CH}{(\theta_2 - \theta'_1 - i\varepsilon)(\theta_1 - \theta'_1 - i\varepsilon)} + \frac{AG + BH}{(\theta_2 - \theta'_1 - i\varepsilon)(\theta_2 - \theta'_2 - i\varepsilon)} + \frac{BE + DG}{(\theta_2 - \theta'_2 - i\varepsilon)(\theta_1 - \theta'_2 - i\varepsilon)} \quad (S75)$$

$$+ \frac{BF + CG}{(\theta_2 - \theta'_2 - i\varepsilon)(\theta_1 - \theta'_1 - i\varepsilon)} + \frac{CE + DF}{(\theta_1 - \theta'_1 - i\varepsilon)(\theta_1 - \theta'_2 - i\varepsilon)} \quad (S76)$$

with

$$\begin{aligned} A &= i[S(\theta_2 - \theta_1) - S(\theta'_1 - \theta'_2)] F_2^\sigma(\theta_1 + i\pi, \theta'_2); \quad B = i[S(\theta'_1 - \theta'_2) S(\theta_2 - \theta_1) - 1] F_2^\sigma(\theta_1 + i\pi, \theta'_1); \\ C &= i[1 - S(\theta_2 - \theta_1) S(\theta'_1 - \theta'_2)] F_2^\sigma(\theta_2 + i\pi, \theta'_2); \quad D = i[S(\theta'_1 - \theta'_2) - S(\theta_2 - \theta_1)] F_2^\sigma(\theta_2 + i\pi, \theta'_1); \\ E &= i[S(\theta_1 - \theta_2) - S(\theta'_2 - \theta'_1)] F_2^\sigma(\theta_2 + i\pi, \theta'_1); \quad F = i[S(\theta'_2 - \theta'_1) S(\theta_1 - \theta_2) - 1] F_2^\sigma(\theta_2 + i\pi, \theta'_2); \\ G &= i[1 - S(\theta_1 - \theta_2) S(\theta'_2 - \theta'_1)] F_2^\sigma(\theta_1 + i\pi, \theta'_1); \quad H = i[S(\theta'_2 - \theta'_1) - S(\theta_1 - \theta_2)] F_2^\sigma(\theta_1 + i\pi, \theta'_2) \end{aligned} \quad (S77)$$

From Eq. (S71) we have (q and ω has been rescaled by Δ_a)

$$\begin{aligned} I_8^{(1)}(\omega, q) &= \frac{1}{\Delta_a^2} \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^2} F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1, \theta'_2) F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi, \theta'_2, \theta'_1) \\ &\delta(q + \sinh \theta_1 + \sinh \theta_2 - \sinh \theta'_1 - \sinh \theta'_2) \delta(\omega + \cosh \theta_1 + \cosh \theta_2 - \cosh \theta'_1 - \cosh \theta'_2) \\ &= \frac{1}{\Delta_a^2} \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \frac{F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1, \theta'_2) F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi, \theta'_2, \theta'_1) e^{-\beta \Delta_a (\cosh \theta_1 + \cosh \theta_2)}}{\left\{ \left[\frac{(\omega + \cosh \theta_1 + \cosh \theta_2)^2 - (q + \sinh \theta_1 + \sinh \theta_2)^2}{2} - 1 \right]^2 - 1 \right\}^{1/2}} \end{aligned} \quad (S78)$$

which leads to

$$I_8^{(1)}(\omega) = \frac{1}{\Delta_a} \int dq \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \frac{F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1, \theta'_2) F_{4rc}^\sigma(\theta_1 + i\pi, \theta_2 + i\pi, \theta'_2, \theta'_1) e^{-\beta \Delta_a (\cosh \theta_1 + \cosh \theta_2)}}{\left\{ \left[\frac{(\omega + \cosh \theta_1 + \cosh \theta_2)^2 - (q + \sinh \theta_1 + \sinh \theta_2)^2}{2} - 1 \right]^2 - 1 \right\}^{1/2}} \quad (S79)$$

where θ'_1 and θ'_2 are functions of θ_1 and θ_2 . Then we can apply steepest descent method on $I_8^{(1)}(\omega)$, leading to (unlike S_{11} , here θ_1 and θ_2 are independent of q and ω),

$$I_8^{(1)}(\omega) \sim \frac{(F_{4rc}^\sigma(i\pi, i\pi|0, 0))^2}{\Delta_a} \frac{T}{\Delta_a} e^{-2\Delta_a/T} \int dq dq \frac{1}{\sqrt{\left[\left(\frac{(\omega+2)^2 - q^2}{2} \right) - 1 \right]^2 - 1}} \quad (S80)$$

The allowed integration range of q can be determined by

$$\begin{aligned} \left[\left(\frac{(\omega+2)^2 - q^2}{2} \right) - 1 \right]^2 - 1 \geq 0 &\Rightarrow (\omega^2 + 4\omega - q^2) \left[\frac{1}{4} (\omega^2 + 4\omega - q^2) + 1 \right] \geq 0 \\ (q^2 - \omega^2 - 4\omega) (q^2 - 4\omega - \omega^2 - 4) &\geq 0 \Rightarrow q^2 \geq 4 + \omega^2 + 4\omega \text{ or } q^2 \leq \omega^2 + 4\omega \end{aligned} \quad (S81)$$

Using evenness of the integrand as a function of q (so the integral over q can be shrunk to $(0, \infty)$) and making variable

transform $z = \omega^2 + 4\omega - q^2$, we have

$$\begin{aligned}
I_8^{(1)}(\omega) &\sim \frac{(F_{4rc}^\sigma(i\pi, i\pi|0, 0))^2}{\pi\Delta_a} \frac{T}{\Delta_a} e^{-2\Delta_a/T} \left(\int_4^\infty dz + \int_{-(4\omega+\omega^2)}^0 dz \right) \int dq \frac{1}{\sqrt{(z+4\omega+\omega^2)(z-4)z}} \\
&= \frac{(F_{4rc}^\sigma(i\pi, i\pi|0, 0))^2}{\pi\Delta_a} \frac{T}{\Delta_a} e^{-2\Delta_a/T} \left[\frac{2(iK(-4/a) + K(1+4/a))}{\sqrt{a}} + K(-a/4) \right] \quad (a = \omega^2 + 4\omega) \\
&= \frac{(F_{4rc}^\sigma(i\pi, i\pi|0, 0))^2}{\pi\Delta_a} \frac{T}{\Delta_a} e^{-2\Delta_a/T} \left\{ \pi - \frac{\pi}{16}a + \frac{9\pi}{1024}a^2 + \dots \right\} \quad (a \ll 1) \\
&= \frac{(F_{4rc}^\sigma(i\pi, i\pi|0, 0))^2}{\Delta_a} \frac{T}{\Delta_a} e^{-2\Delta_a/T} \left\{ 1 - \frac{1}{4} \frac{\omega}{\Delta_a} + \dots \right\}
\end{aligned} \tag{S82}$$

where K is the complete elliptic integral of the first kind. Therefore, $I_8^{(1)}$ is negligible for the low-energy local dynamics compared with S_{11} .

For Eqs. (S72, S73), all terms have a similar structure, so we can just focus on one of them.

$$\begin{aligned}
I_8^{(2)} &= \frac{1}{4} \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1 + i\varepsilon, \theta'_2 + i\varepsilon) \frac{E}{\theta_1 - \theta'_2 - i\varepsilon} \\
&\quad e^{-\beta\Delta_a(\cosh\theta_1 + \cosh\theta_2)} e^{-i\Delta_a x(\sinh\theta_1 + \sinh\theta_2 - \sinh\theta'_1 - \sinh\theta'_2)} e^{-it\Delta_a(\cosh\theta'_1 + \cosh\theta'_2 - \cosh\theta_1 - \cosh\theta_2)} \\
&= I_8^{(2),1} + I_8^{(2),2}
\end{aligned} \tag{S83}$$

where

$$\begin{aligned}
I_8^{(2),1} &= \frac{1}{4} P \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1 + i\varepsilon, \theta'_2 + i\varepsilon) \frac{E}{\theta_1 - \theta'_2} \\
&\quad e^{-\beta\Delta_a(\cosh\theta_1 + \cosh\theta_2)} e^{-i\Delta_a x(\sinh\theta_1 + \sinh\theta_2 - \sinh\theta'_1 - \sinh\theta'_2)} e^{-it\Delta_a(\cosh\theta'_1 + \cosh\theta'_2 - \cosh\theta_1 - \cosh\theta_2)}
\end{aligned} \tag{S84}$$

and

$$\begin{aligned}
I_8^{(2),2} &= \frac{1}{4} i\pi \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1 + i\varepsilon, \theta'_2 + i\varepsilon) E\delta(\theta_1 - \theta'_2) \\
&\quad e^{-\beta\Delta_a(\cosh\theta_1 + \cosh\theta_2)} e^{-i\Delta_a x(\sinh\theta_1 + \sinh\theta_2 - \sinh\theta'_1 - \sinh\theta'_2)} e^{-it\Delta_a(\cosh\theta'_1 + \cosh\theta'_2 - \cosh\theta_1 - \cosh\theta_2)}
\end{aligned} \tag{S85}$$

For $I_8^{(2),1}$, the principal value integral structure will be similar as that appearing in I_7 . Similar analysis can be applied to $I_8^{(2),1}$, leading to a non-singular contribution in the low-frequency regime (it's a four-fold integration similar to that appearing in $I_8^{(1)}$). As for $I_8^{(2),2}$ it's easy to get

$$I_8^{(2),2} = \frac{1}{4} i\pi \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} F_{4rc}^\sigma(\theta_2 + i\pi, \theta_1 + i\pi|\theta'_1, \theta'_2) E|_{\theta_1=\theta'_2} e^{-\beta\Delta_a(\cosh\theta_1 + \cosh\theta_2)} e^{-i\Delta_a x(\sinh\theta_2 - \sinh\theta'_1)} e^{-it\Delta_a(\cosh\theta'_1 - \cosh\theta_2)} \tag{S86}$$

where $E|_{\theta_1=\theta'_2} = i[S(-\theta_2) - S(-\theta'_1)] F_2^\sigma(\theta_2 + i\pi, \theta'_1)$. We encounter similar integral structure as shown in $I_6^{(2)}$. Thus, this part's contribution will be of the same order as that appearing in $I_6^{(2)}$. Therefore the contribution from $I_8^{(2)}$ to the low-energy local dynamics is negligible compared with S_{11} .

For Eqs. (S74, S75, S76), let's first consider the parts containing terms similar to the following:

$$\frac{AE + DH}{(\theta_2 - \theta'_1 - i\varepsilon)(\theta_1 - \theta'_2 - i\varepsilon)} \tag{S87}$$

Other five similar terms will have contribution at the same order of this one. For this one we have

$$\frac{AE + DH}{(\theta_2 - \theta'_1 - i\varepsilon)(\theta_1 - \theta'_2 - i\varepsilon)} = P \frac{1}{\theta_2 - \theta'_1} P \frac{1}{\theta_1 - \theta'_2} (AE + DH) + P \frac{1}{\theta_2 - \theta'_1} i\pi\delta(\theta_1 - \theta'_2) (AE + DH) \tag{S88}$$

$$P \frac{1}{\theta_1 - \theta'_2} i\pi\delta(\theta_2 - \theta'_1) (AE + DH) - \pi^2\delta(\theta_2 - \theta'_1)\delta(\theta_1 - \theta'_2) (AE + DH) \tag{S89}$$

For the first term we will encounter similar structure as $I_8^{(1)}$, and for the second and third terms we will encounter similar structure as $I_8^{(2)}$. It is also easy to determine $\pi^2 \delta(\theta_2 - \theta'_1) \delta(\theta_1 - \theta'_2) (AE + DH) = 0$. Thus, the total contribution from the term containing $\frac{AE+DH}{(\theta_2-\theta'_1-i\varepsilon)(\theta_1-\theta'_2-i\varepsilon)}$ is negligible. This applies to other similar terms, in which there can exist non-vanishing terms of two multiples of delta functions. The terms having this kind of structure will have similar integral structure as S_{11} , after integrating over the two delta functions. But the thermal factor $e^{-2\Delta_a/T}$ makes this negligible.

We next discuss the last terms which have a similar structure as

$$\frac{AH}{(\theta_2 - \theta'_1 - i\varepsilon)^2} \quad (S90)$$

Such terms can formerly be handled as follows,

$$\frac{AH}{(\theta_2 - \theta'_1 - i\varepsilon)^2} \rightarrow \text{Integration by part} \rightarrow \int \frac{1}{\theta'_1 - \theta_2 + i\varepsilon} \partial_{\theta'_1} (AH \dots \dots) \quad (S91)$$

Combining the contributions from four such terms with that appearing in $I_6^{(1)}$ will yield zero contribution to the low-energy local dynamics. Explicitly we have

$$\begin{aligned} AH &= -[S(\theta_2 - \theta_1) - S(\theta'_1 - \theta'_2)] F_2^\sigma(\theta_1 + i\pi, \theta'_2) [S(\theta'_2 - \theta'_1) - S(\theta_1 - \theta_2)] F_2^\sigma(\theta_1 + i\pi, \theta'_2) \\ &= [2 - S(\theta_2 - \theta_1)S(\theta'_2 - \theta'_1) - S(\theta_1 - \theta_2)S(\theta'_1 - \theta'_2)] (F_2^\sigma(\theta_1 + i\pi, \theta'_2))^2 \end{aligned} \quad (S92)$$

\Rightarrow

$$\begin{aligned} &\frac{1}{4} \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} \frac{AH}{(\theta_2 - \theta'_1 - i\varepsilon)^2} K_{tx}^{(\beta)}(\theta_1 \theta_2 | \theta'_1 \theta'_2) \\ &= \frac{1}{4} \int \frac{d\theta_1 d\theta_2 d\theta'_2}{(2\pi)^4} \frac{AH K_{tx}^{(\beta)}(\theta_1 \theta_2 | \theta'_1 \theta'_2)}{\theta_2 - \theta'_1 - i\varepsilon} \Big|_{\theta'_1=-\infty}^{\theta'_1=\infty} + \frac{1}{4} \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} \frac{1}{\theta'_1 - \theta_2 + i\varepsilon} \left[K_{tx}^{(\beta)} \partial_{\theta'_1} (AH) + AH \left(\partial_{\theta'_1} K_{tx}^{(\beta)} \right) \right] \\ &= \frac{1}{4} \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} \left[P \frac{1}{\theta'_1 - \theta_2} - i\pi \delta(\theta'_1 - \theta_2) \right] \left[K_{tx}^{(\beta)} \partial_{\theta'_1} (AH) + AH \left(\partial_{\theta'_1} K_{tx}^{(\beta)} \right) \right] \end{aligned} \quad (S93)$$

where

$$K_{tx}^{(\beta)}(\theta_1 \theta_2 | \theta'_1 \theta'_2) = e^{-\beta \Delta_a (\cosh \theta_1 + \cosh \theta_2)} e^{-i \Delta_a x (\sinh \theta_1 + \sinh \theta_2 - \sinh \theta'_1 - \sinh \theta'_2)} e^{-it \Delta_a (\cosh \theta'_1 + \cosh \theta'_2 - \cosh \theta_1 - \cosh \theta_2)} \quad (S94)$$

Let's focus on the following integral (all other integrals will have similar features as before),

$$\frac{1}{4} \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} [-i\pi \delta(\theta'_1 - \theta_2)] \left[AH \left(\partial_{\theta'_1} K_{tx}^{(\beta)} \right) \right] \quad (S95)$$

where

$$\begin{aligned} [-i\pi \delta(\theta'_1 - \theta_2)] \left[AH \left(\partial_{\theta'_1} K_{tx}^{(\beta)} \right) \right] &= \pi \delta(\theta'_1 - \theta_2) [2 - S(\theta_2 - \theta_1)S(\theta'_2 - \theta_2) - S(\theta_1 - \theta_2)S(\theta_2 - \theta'_2)] \\ &\quad (F_2^\sigma(\theta_1 + i\pi, \theta'_2))^2 (x \Delta_a \cosh \theta_2 - t \Delta_a \sinh \theta_2) K_{tx}^{(\beta)}(\theta_1 \theta_2 | \theta_2 \theta'_2) \end{aligned} \quad (S96)$$

Substituting the above results back into the integral, and after finishing the integration over the delta function we can re-label the integral variables as follows

$$\theta_1 \leftrightarrow \theta_2 \text{ and } \theta'_2 \rightarrow \theta'_1 \quad (S97)$$

we get

$$\begin{aligned} &\frac{1}{4} \int \frac{d\theta_1 d\theta_2 d\theta'_1 d\theta'_2}{(2\pi)^4} [-i\pi \delta(\theta'_1 - \theta_2)] \left[AH \left(\partial_{\theta'_1} K_{tx}^{(\beta)} \right) \right] = \\ &\frac{1}{8} \int \frac{d\theta_1 d\theta_2 d\theta'_1}{(2\pi)^3} [2 - S(\theta_1 - \theta_2)S(\theta'_1 - \theta_1) - S(\theta_2 - \theta_1)S(\theta_1 - \theta'_1)] \\ &\quad (F_2^\sigma(\theta_1 + i\pi, \theta'_2))^2 (x \Delta_a \cosh \theta_1 - t \Delta_a \sinh \theta_1) K_{tx}^{(\beta)}(\theta_1 \theta_2 | \theta_1 \theta'_1) \end{aligned} \quad (S98)$$

For the other three similar terms, one can get a similar integral as above for the part we are interested in. These parts can be combined with that appearing in $I_6^{(1)}$ and yield

$$I_c(x, t) \equiv I_8^{part} + I_6^{(1)} = \frac{1}{2} \int \frac{d\theta_1 d\theta_2 d\theta'_1}{(2\pi)^3} [S(\theta_1 - \theta_2)S(\theta'_1 - \theta_1) - S(\theta_2 - \theta_1)S(\theta_1 - \theta'_1)] \\ (F_2^\sigma(\theta_1 + i\pi, \theta'_2))^2 (x\Delta_a \cosh \theta_1 - t\Delta_a \sinh \theta_1) K_{tx}^{(\beta)}(\theta_1 \theta_2 | \theta_1 \theta'_1) \quad (\text{S99})$$

\Rightarrow

$$I_c(\omega) = \frac{1}{2} \int \frac{d\theta_1 d\theta_2 d\theta'_1}{(2\pi)^3} u(\theta_1, \theta_2, \theta'_1, \omega) \quad (\text{S100})$$

with

$$u(\theta_1, \theta_2, \theta'_1, \omega) = (F_2^\sigma(\theta_1 + i\pi, \theta'_2))^2 e^{-\beta\Delta_a(\cosh \theta_1 + \cosh \theta_2)} e^{-it\Delta_a(\cosh \theta'_1 - \cosh \theta_2)} \cdot [S(\theta_1 - \theta_2)S(\theta'_1 - \theta_1) - S(\theta_2 - \theta_1)S(\theta_1 - \theta'_1)] \frac{2\pi\delta[\omega - \Delta_a(\cosh \theta'_1 - \cosh \theta_2)]}{\omega - \Delta_a(\cosh \theta'_1 - \cosh \theta_2)} \quad (\text{S101})$$

Because

$$u(-\theta_1, -\theta_2, -\theta'_1, \omega) = -u(\theta_1, \theta_2, \theta'_1, \omega), \quad (\text{S102})$$

we have $I_c(\omega) = 0$.

Combining all of the above, we conclude that (except for the time-independent parts in I_8 , see below) there are no singularities in the frequency dependence that are stronger than that of S_{11} , and the thermal factor $e^{-2\Delta_a/T}$ makes S_{22} to be negligible compared to S_{11} .

Disconnected Contributions up to D_{22}

At $x \rightarrow \infty$ we expect the following cluster property,

$$\langle \sigma(x, t) \sigma(0, 0) \rangle_T \sim \langle \sigma(0, 0) \rangle_T^2 \quad (\text{S103})$$

Applying the Leclair-Mussardo formula [20] for the single-point function $\langle \sigma(0, 0) \rangle_T$ in Eq. (S103), we can get the part which contributes time-independent pieces in the two-point correlation function $\langle \sigma(x, t) \sigma(0, 0) \rangle_T$. Indeed in the E_8 model, up to $e^{-3\Delta_i/T}$ ($i = a, b, c$), the time independent parts up to D_{22} can be summed over to $\langle \sigma \rangle_{T,i}^2 + O(e^{-3\Delta_i/T})$ with

$$\langle \sigma \rangle_{T,i} = \langle \sigma \rangle_0 + \int \frac{d\theta_1}{2\pi} F_2^\sigma(i\pi, 0) e^{-\beta\Delta_i \cosh \theta_1} - \int \frac{d\theta_1}{2\pi} F_2^\sigma(i\pi, 0) e^{-2\beta\Delta_i \cosh \theta_1} \\ + \frac{1}{2} \int \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} F_{4s}^\sigma(\theta_1, \theta_2) e^{-\beta\Delta_i(\cosh \theta_1 + \cosh \theta_2)} + O(e^{-3\Delta_i/T}) \quad (i = a, b, c) \quad (\text{S104})$$

It's easy to see that the expressions above for $\langle \sigma \rangle_{T,i}$ correspond term-by-term to Leclair-Mussardo formula [20]. We thus expect that, when summing over to infinite terms of the expansion series, the contribution from all of these space-time independent terms will sum over to $\langle \sigma(0, 0) \rangle_T^2$. In other words, none of the time-independent terms in the two-point correlation function will appear in the two-point connected correlation function.